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Controllability of series connections of arbitrarily many linear systems ☆

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Abstract

In this paper we study the controllability of series connections of arbitrarily many linear systems. As the main result, we completely determine the controllability and the possible controllability indices of a system obtained by a special series connection of arbitrarily many linear systems.

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1. Introduction

Let $S_1, \ldots, S_n$ be linear systems described by the following systems of ordinary differential equations of the first degree:

$$S_i \begin{cases} \dot{x}_i = A_i x_i + B_i u_i \\ y_i = C_i x_i \end{cases} \quad i = 1, \ldots, n, \tag{1}$$

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where $A_i \in \mathbb{K}^{n_i \times n_i}$, $B_i \in \mathbb{K}^{n_i \times m_i}$, $C_i \in \mathbb{K}^{p_i \times n_i}$, $i = 1, \ldots, n$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, while u_i, y_i, and x_i are the input, the output and the state, respectively, of S_i, $i = 1, \ldots, n$, for details see [4].

Since algebraic properties of the system S_i depend only on the properties of the matrices A_i, B_i and C_i, $i = 1, \ldots, n$, we can consider them over an arbitrary field $\mathbb{F}$. Also, recall that the system S_i is *controllable* if and only if the pair (A_i, B_i) is *controllable*:

Definition 1. Let $A \in \mathbb{F}^{n \times n}$, $B \in \mathbb{F}^{n \times m}$. The pair (A, B) is controllable if and only if one of the following (equivalent) conditions is satisfied:

(1) $\min_{\lambda \in \mathbb{F}} \operatorname{rank} \begin{bmatrix} \lambda I - A & -B \end{bmatrix} = n$,
(2) all invariant factors of the matrix pencil $\begin{bmatrix} \lambda I - A & -B \end{bmatrix}$ are trivial,
(3) $\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$.

In this case we also say that the corresponding matrix $[A \quad B]$ and the pencil $[\lambda I - A \quad -B]$ are controllable.

In this paper we study series connections of more than two linear systems. In fact, we study two different types of connections of $n \geqslant 3$ linear systems, described by the following equations

$$u_{i+1} = \overline{X}_i y_i, \quad \text{where } \overline{X}_i \in \mathbb{F}^{m_{i+1} \times p_i},\ i = 1, \ldots, n-1, \tag{2}$$

and

$$u_j = \sum_{i=2}^{j} \overline{X}_{j-1\,i-1} y_{i-1}, \quad \text{where } \overline{X}_{j-1\,i-1} \in \mathbb{F}^{m_j \times p_{i-1}},\ 2 \leqslant i \leqslant j \leqslant n. \tag{3}$$

As a result of these connections, we obtain a new system S, with input u_1, output y_n and state $[x_1^{\mathrm{T}} \quad \cdots \quad x_n^{\mathrm{T}}]^{\mathrm{T}}$.

The connections of the type (2) are further called *the series connections of the first type*, and for them the resulting system is controllable if and only if the matrix

$$\left[\begin{array}{cccccc|c} A_1 & 0 & 0 & \cdots & 0 & B_1 \\ B_2\overline{X}_1C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3\overline{X}_2C_2 & A_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & B_n\overline{X}_{n-1}C_{n-1} & A_n & 0 \end{array}\right], \tag{4}$$

is controllable.

Analogously, the connections of the type (3) are further called *the series connections of the second type*, and for them the resulting system is controllable if and only if the matrix

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2\overline{X}_{11}C_1 & A_2 & 0 & \ddots & 0 & 0 \\ B_3\overline{X}_{21}C_1 & B_3\overline{X}_{22}C_2 & A_3 & \ddots & \vdots & \vdots \\ \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ B_n\overline{X}_{n-1\,1}C_1 & B_n\overline{X}_{n-1\,2}C_2 & \ddots & B_n\overline{X}_{n-1\,n-1}C_{n-1} & A_n & 0 \end{array}\right], \tag{5}$$

is controllable.

In fact, series connections of the first type describe connections of the linear systems $S_1, \dots, S_n$, such that the input of S_{i+1} is a linear function of the output of S_i, $i = 1, \dots, n-1$.

On the other hand, series connections of the second type describe connections of the linear systems $S_1, \dots, S_n$, such that the input of S_{i+1} is a linear function of the outputs of $S_1, \dots, S_i$, $i = 1, \dots, n-1$.

Necessary and sufficient conditions for the existence of matrices $\overline{X}_i, i = 1, \dots, n-1$, such that the matrix (4) is controllable, are given in [3]. However, the problems of determining the possible controllability indices of the matrix (4), respectively (5), when matrices $\overline{X}_i$, $i = 1, \dots, n-1$, respectively $\overline{X}_{ij}$, $1 \leqslant j \leqslant i \leqslant n-1$, vary, are very difficult and remain open. For some partial results see [1,8]. In order to generalize and extend these results, we have considered only linear systems $S_1, \dots, S_n$, with the properties rank $B_i = n_i$, $i = 2, \dots, n$, and rank $C_i = n_i$, $i = 1, \dots, n-1$. In this case, the problem of determining the possible controllability indices of the matrix (4), when matrices $\overline{X}_i, i = 1, \dots, n-1$, vary, is equivalent to the problem of determining the possible controllability indices of the matrix

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \cdots & 0 & B_1 \\ X_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & X_2 & A_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & X_{n-1} & A_n & 0 \end{array}\right], \tag{6}$$

when matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$, $i = 1, \dots, n-1$, vary.

Analogously, the problem of determining the possible controllability indices of the matrix (5), when matrices $\overline{X}_{ij}$, $1 \leqslant j \leqslant i \leqslant n-1$, vary, is equivalent to the problem of determining the possible controllability indices of the matrix

$$\begin{bmatrix} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ X_{11} & A_2 & 0 & \ddots & 0 & 0 \\ X_{21} & X_{22} & A_3 & \ddots & \vdots & \vdots \\ \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ X_{n-1\,1} & X_{n-1\,2} & \ddots & X_{n-1\,n-1} & A_n & 0 \end{bmatrix}, \tag{7}$$

when matrices $X_{ij} \in \mathbb{F}^{n_{i+1} \times n_j}$, $1 \leqslant j \leqslant i \leqslant n-1$, vary.

In this paper we solve the following problems:

Problem 1. Let $\mathbb{F}$ be an algebraically closed field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \dots, n$ and $B_1 \in \mathbb{F}^{n_1 \times m_1}$. Find necessary and sufficient conditions for the existence of matrices $X_{ij} \in \mathbb{F}^{n_{i+1} \times n_j}$, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix (7) has prescribed controllability indices.

Problem 2. Let $\mathbb{F}$ be an algebraically closed field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \dots, n$ and $B_1 \in \mathbb{F}^{n_1 \times m_1}$. Let (A_1, B_1) be a controllable pair of matrices. Find sufficient conditions for the existence of matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$, $i = 1, \dots, n-1$, such that the matrix (6) is controllable with prescribed controllability indices.

2. Notation

Let $\mathbb{F}$ be a field. For any polynomial f, $d(f)$ denotes its degree. If $f(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_1\lambda - a_0 \in \mathbb{F}[\lambda]$, where $k > 0$, then the matrix

$$\begin{bmatrix} e_2^{(k)} & \cdots & e_k^{(k)} & a \end{bmatrix}^{\mathrm{T}},$$

where $e_i^{(k)}$ is the ith column of the identity matrix I_k and $a = [a_0 \ \cdots \ a_{k-1}]^{\mathrm{T}}$, is called *the companion matrix* for the polynomial $f(\lambda)$.

Definition 2. Two matrices

$$M = \begin{bmatrix} A & B \end{bmatrix}, \qquad M' = \begin{bmatrix} A' & B' \end{bmatrix}, \tag{8}$$

where $A, A' \in \mathbb{F}^{n\times n}$, $B, B' \in \mathbb{F}^{n\times m}$, are said to be feedback equivalent if there exists a nonsingular matrix

$$P = \begin{bmatrix} N & 0 \\ V & T \end{bmatrix}$$

where $N \in \mathbb{F}^{n\times n}$, $V \in \mathbb{F}^{m\times n}$, $T \in \mathbb{F}^{m\times m}$, such that $M' = N^{-1}MP$.

If M and M' are feedback equivalent then we also say that the corresponding pairs (A, B) and (A', B') are feedback equivalent.

It is easy to verify that two matrices of the form (8) are feedback equivalent if and only if the matrix pencils

$$R = \begin{bmatrix} \lambda I - A & -B \end{bmatrix} \quad \text{and} \quad R' = \begin{bmatrix} \lambda I - A' & -B' \end{bmatrix} \tag{9}$$

are strictly equivalent, for details see [5].

Further on in this paper, the concept of majorization in the sense of Hardy–Littlewood–Pólya [6] is used. Given two sequences $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n$, we say that $a = (a_1, \ldots, a_n)$ is *majorized* by $b = (b_1, \ldots, b_n)$ and write $a \prec b$ if

$$\sum_{i=1}^{j} a_{(i)} \leqslant \sum_{i=1}^{j} b_{(i)}, \quad 1 \leqslant j \leqslant n-1,$$
$$\sum_{i=1}^{n} a_{(i)} = \sum_{i=1}^{n} b_{(i)},$$

where $a_{(1)} \geqslant \cdots \geqslant a_{(n)}$ and $b_{(1)} \geqslant \cdots \geqslant b_{(n)}$ are the elements of a and b, respectively, both in nonincreasing order.

Also, we say that the partition a is obtained from the partition b by an elementary transformation whenever there exist indices j and k with $j < k$, such that $a_{(j)} = b_{(j)} - 1$, $a_{(k)} = b_{(k)} + 1$, and $a_{(i)} = b_{(i)}$ for $i \neq j, k$.

Furthermore, we have that

$$\left(a_1, \ldots, a_n\right) \prec \left(b_1, \ldots, b_n\right)$$

if and only if $(a_{(1)}, \ldots, a_{(n)})$ is obtained from $(b_{(1)}, \ldots, b_{(n)})$ by successive applications of a finite number of elementary transformations.

By $S(A, B)$ we denote the controllability matrix of the pair (A, B) i.e.

$$S(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{F}^{n \times nm}.$$

If rank $S(A, B) = r$, then select from left to right the first r linearly independent columns of $S(A, B)$. Write these columns as

$$\{b_1, Ab_1, \ldots, A^{c_1-1}b_1, b_2, Ab_2, \ldots, A^{c_2-1}b_2, \ldots, A^{c_m-1}b_m\},$$

where $c_i = 0$ if b_i is absent. Obviously rank $B = \sharp\{i | c_i > 0\}$. The positive integers among $c_1, c_2, \ldots, c_m$, ordered in nonincreasing order, are called *the* (*nonzero*) *controllability indices* of the pair (A, B) and of the corresponding matrix $[A \quad B]$.

Moreover, the controllability indices of the matrix

$$\begin{bmatrix} A & B \end{bmatrix} \tag{10}$$

coincide with the column minimal indices of the corresponding matrix pencil

$$\begin{bmatrix} \lambda I - A & -B \end{bmatrix}. \tag{11}$$

Further on, we shall consider only the nonzero controllability indices.

3. Auxiliary results

Recall the definition of *the Brunovsky canonical form* of a controllable matrix pair:

If $(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}$ is controllable and $k_1 \geqslant \cdots \geqslant k_s > 0$ are its controllability indices then (A, B) is feedback equivalent to a matrix pair (A_c, B_c) such that

$$A_c = \operatorname{diag}(A_1, \ldots, A_s), \qquad B_c = \begin{bmatrix} \operatorname{diag}(B_1, \ldots, B_s) & 0 \end{bmatrix},$$

where

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{k_i \times k_i}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{F}^{k_i \times 1}, \quad 1 \leqslant i \leqslant s.$$

Lemma 1 (Lemma 3.4 in [1,2]). *Let* $(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}$ *and assume that* rank $S(A, B) = r$. *Then there exists a nonsingular matrix* $P \in \mathbb{F}^{n \times n}$ *such that*

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$(A_1, B_1) \in \mathbb{F}^{r \times r} \times \mathbb{F}^{r \times m}$ *being a controllable pair. A pair* (PAP^{-1}, PB) *is called a Kalman decomposition of* (A, B). *Analogously, we say that* $[PAP^{-1} \quad PB]$ *is a Kalman decomposition of* $[A \quad B]$.

Lemma 2 (Lemma 4.2 in [1,7]). *Let* $\alpha_1 | \cdots | \alpha_n$ *and* $\delta_1, \ldots, \delta_n$ *be* $2n$ *monic polynomials. Then there exists an* $n \times n$ *triangular polynomial matrix with diagonal* $(\delta_1, \ldots, \delta_n)$ *and* $\alpha_1 | \cdots | \alpha_n$ *as invariant factors if and only if*

$$\alpha_1 \cdots \alpha_k \,|\, \gcd\{\delta_{i_1} \cdots \delta_{i_k}, 1 \leqslant i_1 < \cdots < i_k \leqslant n\}, \quad 1 \leqslant k \leqslant n-1,$$
$$\alpha_1 \cdots \alpha_n = \delta_1 \cdots \delta_n.$$

Lemma 3 (Lemma 4.3 in [1,9,10]). *Let $\mathbb{F}$ be an algebraically closed field and let $h_1, \ldots, h_m$ and $\alpha_1 | \cdots | \alpha_m$ be nonnegative integers and monic polynomials, respectively. If*

$$(h_1, \ldots, h_m) \prec (d(\alpha_m), \ldots, d(\alpha_1)),$$

then there exist monic polynomials $\delta_1, \ldots, \delta_m$ such that $d(\delta_i) = h_i$, $1 \leqslant i \leqslant m$ and

$$\alpha_1 \cdots \alpha_k \,|\, \gcd\{\delta_{i1} \cdots \delta_{ik} : 1 \leqslant i_1 < \cdots < i_k \leqslant m\}, \quad 1 \leqslant k \leqslant m-1,$$
$$\alpha_1 \cdots \alpha_m = \delta_1 \cdots \delta_m.$$

In [1] the following result has been proved:

Theorem 3 (Theorem 4.1 and Theorem 4.4 in [1]). *Let $\mathbb{F}$ be an algebraically closed field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, 2$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$. Let (A_1, B_1) be a controllable pair, with $c_1 \geqslant \cdots \geqslant c_s > 0$ as controllability indices. Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers, $\sum_{i=1}^{s} k_i = n_1 + n_2$. There exists a matrix X_1, such that*

$$\begin{bmatrix} A_1 & 0 & B_1 \\ X_1 & A_2 & 0 \end{bmatrix}$$

is controllable, and has $k_1 \geqslant \cdots \geqslant k_s$ as the controllability indices, if and only if the following conditions hold:

(i) $k_i \geqslant c_i, \quad i = 1, \ldots, s,$

(ii) $(k_1 - c_1, \ldots, k_s - c_s) \prec (d(\alpha_{n_2}), \ldots, d(\alpha_{n_2-s+1})),$

where $\alpha_1 | \cdots | \alpha_{n_2}$, are the invariant factors of $\lambda I - A_2$ and $s = \operatorname{rank} B_1$.

Note that if r is the number of nontrivial invariant factors of $\lambda I - A_2$, then from the condition (ii) follows:

$$s \geqslant r.$$

Our aim is to prolong this result for connections of three or more linear systems. In order to do that we need two technical lemmas from the following section.

4. Technical lemmas

Lemma 4. *Let m_j^i, c_j, $j = 1, \ldots, s$, $i = 1, \ldots, t$, be positive integers. Let n_i, $i = 1, \ldots, t$, be nonnegative integers, $n := \sum_{i=1}^{t} n_i$. Let $p = s + \max\{n_1, \ldots, n_t\}$. Let l_i, $i = 1, \ldots, t$, be nonnegative integers, such that $l_i \geqslant p$ for every $i = 1, \ldots, t$. Let A_k^i, $k = 1, \ldots, l_i$, $i = 1, \ldots, t$, be nonnegative integers such that $A_1^i \geqslant \cdots \geqslant A_{l_i}^i$ for each $i = 1, \ldots, t$, and such that*

$$\sum_{j=1}^{s} (m_j^i - m_j^{i-1}) + n_i = \sum_{j=1}^{l_i} A_j^i, \quad i = 1, \ldots, t,$$

where $m_j^0 = c_j$, $j = 1, \ldots, s$. Let $m_j^i \geqslant m_{j'}^i$, $m_j^i \geqslant m_j^{i'} \geqslant c_j$, for all $1 \leqslant i' \leqslant i \leqslant t$ and $1 \leqslant j \leqslant j' \leqslant s$ and let

$$\left(m_1^1 - c_1, \ldots, m_s^1 - c_s, \underbrace{1, \ldots, 1}_{n_1} \right) \prec (A_1^1, \ldots, A_{s+n_1}^1), \tag{12}$$

$$\left(m_1^2 - m_1^1, \ldots, m_s^2 - m_s^1, \underbrace{1, \ldots, 1}_{n_2}\right) \prec (A_1^2, \ldots, A_{s+n_2}^2), \tag{13}$$

$$\vdots$$

$$\left(m_1^t - m_1^{t-1}, \ldots, m_s^t - m_s^{t-1}, \underbrace{1, \ldots, 1}_{n_t}\right) \prec (A_1^t, \ldots, A_{s+n_t}^t). \tag{14}$$

Then

$$\left(m_1^t - c_1, \ldots, m_s^t - c_s, \underbrace{1, \ldots, 1}_{n}\right) \prec \left(\sum_{i=1}^t A_1^i, \ldots, \sum_{i=1}^t A_p^i, \underbrace{0, \ldots, 0}_{s+n-p}\right). \tag{15}$$

Proof. From (12)–(14) we obtain

$$\left.\begin{array}{l} \sum_{i=1}^s (m_i^1 - c_i) + n_1 = A_1^1 + \cdots + A_{s+n_1}^1 \\ \cdots \\ \sum_{i=1}^s (m_i^t - m_i^{t-1}) + n_t = A_1^t + \cdots + A_{s+n_t}^t \end{array}\right\}$$

$$\Rightarrow \sum_{i=1}^s (m_i^t - c_i) + n_1 + \cdots + n_t = \sum_{i=1}^t A_1^i + \cdots + \sum_{i=1}^t A_p^i. \tag{16}$$

Now, for any $i = 1, \ldots, s$, we have

$$m_i^t - c_i = m_i^t - m_i^{t-1} + m_i^{t-1} \cdots + m_i^1 - c_i \leqslant \sum_{i=1}^t A_1^i,$$

so the same is true for $\max_{i=1,\ldots,s}\{m_i^t - c_i\}$. Furthermore, if $i \neq j$,

$$m_i^t - c_i + m_j^t - c_j \leqslant \sum_{i=1}^t A_1^i + \sum_{i=1}^t A_2^i.$$

Hence,

$$\max_{i,j=1,\ldots,s i\neq j} \{m_i^t - c_i + m_j^t - c_j\} \leqslant \sum_{i=1}^t A_1^i + \sum_{i=1}^t A_2^i.$$

Analogously, we obtain

$$\max\{m_{i_1}^t - c_{i_1} + \cdots + m_{i_j}^t - c_{i_j}\} \leqslant \sum_{i=1}^t A_1^i + \cdots + \sum_{i=1}^t A_j^i,$$

for $1 \leqslant j \leqslant s$, where the maximum is taken over all subsets $\{i_1, \ldots, i_j\} \subset \{1, \ldots, s\}$ of j elements.

By using the property of majorization, we can assume that $m_i^t - c_i \geqslant 1$, $i = 1, \ldots, s$. Let $s + n_1 + \cdots + n_t \geqslant j > s$. Now, we have

$$
\begin{aligned}
& m_1^t - c_1 + \cdots + m_s^t - c_s + \underbrace{1 + \cdots + 1}_{j-s} \\
& \leqslant m_1^t - c_1 + \cdots + m_s^t - c_s + \sharp\{i|n_i \geqslant 1\} + \sharp\{i|n_i \geqslant 2\} + \cdots + \sharp\{i|n_i \geqslant j-s\} \\
& = m_1^t - m_1^{t-1} + m_1^{t-1} + \cdots + m_1^1 - c_1 + m_2^t - m_2^{t-1} + m_2^{t-1} \\
& \quad + \cdots + m_2^1 - c_2 + \cdots + m_s^t - m_s^{t-1} + m_s^{t-1} + \cdots + m_s^1 - c_s \\
& \quad + \sharp\{i|n_i \geqslant 1\} + \cdots + \sharp\{i|n_i \geqslant j-s\} \leqslant \sum_{i=1}^t A_1^i + \cdots + \sum_{i=1}^t A_j^i.
\end{aligned}
$$

The last inequality follows from (12)–(14). This together with (16) concludes our proof. $\square$

Lemma 5. *Let $k_1 \geqslant \cdots \geqslant k_s > 0$ and $c_1 \geqslant \cdots \geqslant c_s > 0$ be nonincreasing sequences of positive integers, $k_i \geqslant c_i$, $i = 1, \ldots, s$. Let $A_1 \geqslant \cdots \geqslant A_s \geqslant 0$ and $B_1 \geqslant \cdots \geqslant B_s \geqslant 0$ be nonincreasing sequences of nonnegative integers such that*

$$(k_1 - c_1, \ldots, k_s - c_s) \prec (A_1 + B_1, \ldots, A_s + B_s). \tag{17}$$

Then there exists a nonincreasing sequence $f_1 \geqslant \cdots \geqslant f_s > 0$ of positive integers such that

$$k_i \geqslant f_i \geqslant c_i, \quad i = 1, \ldots, s, \tag{18}$$

and such that

$$(f_1 - c_1, \ldots, f_s - c_s) \prec (A_1, \ldots, A_s), \tag{19}$$

$$(k_1 - f_1, \ldots, k_s - f_s) \prec (B_1, \ldots, B_s). \tag{20}$$

Proof. We shall define $f_1 \geqslant \cdots \geqslant f_s$ by induction on the number of elementary operations by which partitions from (17) differ. Denote this number by n.

The base of induction is the case $n = 0$. In this case

$$k_i - c_i = A_{\sigma(i)} + B_{\sigma(i)}$$

for some permutation σ of $\{1, \ldots, s\}$, and for all $i = 1, \ldots, s$. In particular, we define $f_1, \ldots, f_s$ in the following way

$$f_i := c_i + A_{\sigma(i)} = k_i - B_{\sigma(i)}, \quad i = 1, \ldots, s.$$

Now, it is easy to see that

$$(f_1 - c_1, \ldots, f_s - c_s) = (A_{\sigma(1)}, \ldots, A_{\sigma(s)}) \prec (A_1, \ldots, A_s)$$

and

$$(k_1 - f_1, \ldots, k_s - f_s) = (B_{\sigma(1)}, \ldots, B_{\sigma(s)}) \prec (B_1, \ldots, B_s).$$

Also,

$$k_i \geqslant f_i \geqslant c_i, \quad i = 1, \ldots, s.$$

Thus, we are left with proving that $f_1, \ldots, f_s$ are nonincreasing.

Let $i \geqslant j$, $i, j = 1, \ldots, s$, then $c_i \leqslant c_j$.

If $A_{\sigma(i)} \leqslant A_{\sigma(j)}$ then $f_i = c_i + A_{\sigma(i)} \leqslant c_j + A_{\sigma(j)} = f_j$, as wanted.

So, we are left with the case $A_{\sigma(i)} > A_{\sigma(j)}$. Then $\sigma(i) < \sigma(j)$, and hence $B_{\sigma(j)} \leqslant B_{\sigma(i)}$. Finally, from $k_i \leqslant k_j$, we obtain

$$A_{\sigma(i)} + B_{\sigma(i)} + c_i = k_i \leqslant k_j = A_{\sigma(j)} + B_{\sigma(j)} + c_j,$$

i.e.,

$$f_i - f_j \leqslant B_{\sigma(j)} - B_{\sigma(i)} \leqslant 0,$$

as wanted.

Now we go to a general case. Suppose that the wanted $f_1 \geqslant \cdots \geqslant f_s$ exist if the partitions from (17) differ by $n-1$ elementary operations. Our aim is to prove that the wanted $f_1 \geqslant \cdots \geqslant f_s$ exist if the partitions from (17) differ by n elementary operations. Let π be a permutation of the set $\{1, \ldots, s\}$, such that $k_{\pi(i)} - c_{\pi(i)}, i = 1, \ldots, s$, are in nonincreasing order, i.e., such that $k_{\pi(1)} - c_{\pi(1)} \geqslant \cdots \geqslant k_{\pi(s)} - c_{\pi(s)}$. Let $r := \min\{i | A_i + B_i > k_{\pi(i)} - c_{\pi(i)}\}$ and $r' := \max\{i | A_i + B_i < k_{\pi(i)} - c_{\pi(i)}\}$. Let $l := \max\{i | A_i + B_i = A_r + B_r\}$ and $l' := \min\{i | A_i + B_i = A_{r'} + B_{r'}\}$. Let $(C_1, \ldots, C_s)$ be a partition defined as

$$C_i := A_i + B_i, \quad i \neq l, l', \tag{21}$$

$$C_l := A_l + B_l - 1, \tag{22}$$

$$C_{l'} := A_{l'} + B_{l'} + 1. \tag{23}$$

From the definition of l and l' we have that $A_l + B_l - 1 \geqslant A_{l'} + B_{l'} + 1$, and thus, $l < l'$. Hence, at least one of the following two inequalities is valid:

$$A_l > A_{l'} \quad \text{or} \quad B_l > B_{l'}.$$

Obviously, the partition $(C_1, \ldots, C_s)$ differs from the partition $(k_1 - c_1, \ldots, k_s - c_s)$ by $n-1$ elementary operations, and from the partition $(A_1 + B_1, \ldots, A_s + B_s)$ by one elementary operation. Furthermore, from the definition of l and l', there exist two partitions $(\widetilde{A}_1, \ldots, \widetilde{A}_s)$ and $(\widetilde{B}_1, \ldots, \widetilde{B}_s)$ satisfying

$$\widetilde{A}_1 \geqslant \cdots \geqslant \widetilde{A}_s, \quad \widetilde{B}_1 \geqslant \cdots \geqslant \widetilde{B}_s,$$

such that

$$\widetilde{A}_i + \widetilde{B}_i = C_i, \quad i = 1, \ldots, s,$$

and

$$\widetilde{A}_i = A_i \quad \text{and} \quad \widetilde{B}_i = B_i, \quad i \neq l, l',$$

so that at the positions l and l' one of the following cases is valid:

$$\widetilde{A}_l = A_l - 1, \quad \widetilde{A}_{l'} = A_{l'} + 1, \quad \widetilde{B}_{l'} = B_{l'}, \quad \widetilde{B}_l = B_l, \tag{24}$$

$$\widetilde{B}_l = B_l - 1, \quad \widetilde{B}_{l'} = B_{l'} + 1, \quad \widetilde{A}_{l'} = A_{l'}, \quad \widetilde{A}_l = A_l, \tag{25}$$

$$\widetilde{A}_l = A_l - 1, \quad \widetilde{B}_{l'} = B_{l'} + 1, \quad \widetilde{A}_{l'} = A_{l'}, \quad \widetilde{B}_l = B_l, \tag{26}$$

$$\widetilde{B}_l = B_l - 1, \quad \widetilde{A}_{l'} = A_{l'} + 1, \quad \widetilde{A}_l = A_l, \quad \widetilde{B}_{l'} = B_{l'}. \tag{27}$$

Now, we can apply the induction hypothesis and conclude the existence of the positive integers $\tilde{f}_1 \geqslant \cdots \geqslant \tilde{f}_s$, satisfying

$$k_i \geqslant \tilde{f}_i \geqslant c_i, \quad i = 1, \ldots, s,$$
$$(\tilde{f}_1 - c_1, \ldots, \tilde{f}_s - c_s) \prec (\widetilde{A}_1, \ldots, \widetilde{A}_s),$$
$$(k_1 - \tilde{f}_1, \ldots, k_s - \tilde{f}_s) \prec (\widetilde{B}_1, \ldots, \widetilde{B}_s).$$

We are left to show that in each of the cases (24)–(27), we can define $f_1, \ldots, f_s$ as required in the lemma.

Suppose that (24) is satisfied. Then define

$$f_i := \tilde{f}_i, \quad i = 1, \ldots, s. \tag{28}$$

Such defined $f_1, \ldots, f_s$ are nonincreasing, satisfying $k_i \geqslant f_i \geqslant c_i$, $i = 1, \ldots, s$, and

$$(f_1 - c_1, \ldots, f_s - c_s) \prec (\widetilde{A}_1, \ldots, \widetilde{A}_s) \prec (A_1, \ldots, A_s).$$

The last is true since in the case (24), we have $A_l > A_{l'}$ which implies $(\widetilde{A}_l, \widetilde{A}_{l'}) \prec (A_l, A_{l'})$.

The case (25) is completely analogous to the previous one, so define $f_1, \ldots, f_s$ as in (28).

Since the cases (26) and (27) are symmetric, we shall give the solution only for the case (26), and the solution for the case (27) follows analogously.

Thus, suppose that (26) is satisfied. Our aim is to prove that there exists an index $m \in \{1, \ldots, s\}$ such that if we define $f_1, \ldots, f_s$ as

$$f_i := \tilde{f}_i, \quad i = 1, \ldots, s, \quad i \neq m, \tag{29}$$
$$f_m := \tilde{f}_m + 1, \tag{30}$$

they satisfy (18)–(20).

Let $x_1 \geqslant \cdots \geqslant x_s \geqslant 0$ and $y_1 \geqslant \cdots \geqslant y_s \geqslant 0$ be such that

$$(x_1, \ldots, x_s) \prec (y_1, \ldots, y_s).$$

Let $l, l' \in \{1, \ldots, s\}$. Then

$$(x_1, \ldots, x_i + 1, \ldots, x_s) \prec (y_1, \ldots, y_l + 1, \ldots, y_s),$$
$$(x_1, \ldots, x_j - 1, \ldots, x_s) \prec (y_1, \ldots, y_{l'} - 1, \ldots, y_s),$$

for $i \geqslant l$ and $j \leqslant l'$, $i, j = 1, \ldots, s$.

Thus, if $l = \max\{i | A_i + B_i = A_r + B_r\}$ and $l' = \min\{i | A_i + B_i = A_{r'} + B_{r'}\}$, in order (19) and (20) to be valid, m can be any index from 1 to s, except some $l - 1$ indices (of the biggest $l - 1$ among $(\tilde{f}_i - c_i)$'s) and some $s - l'$ indices (of the smallest $s - l'$ among $(k_i - \tilde{f}_i)$'s). Denote the remaining set of indices by S' (S' is nonempty since $l < l' \Rightarrow l - 1 + s - l' < s$).

In order (18) to be valid, we need to prove that there exists $m \in S'$ such that $\tilde{f}_m < k_m$. Let $i \in \{1, \ldots, s\}$ be such that $\tilde{f}_i = k_i$. Since $\widetilde{B}_{l'} > 0$, we have that $k_i - \tilde{f}_i$ is among the smallest $s - l'$ of $(k_j - \tilde{f}_j)$'s, and hence $i \notin S'$. Thus for all the indices $m \in S'$, $\tilde{f}_m < k_m$.

Finally, in order $f_1, \ldots, f_s$ to be nonincreasing, we need to prove that there exists an index $m \in S'$ such that $f_{m-1} \geqslant f_m \geqslant f_{m+1}$, i.e., $\tilde{f}_{m-1} \geqslant \tilde{f}_m + 1 \geqslant \tilde{f}_{m+1}$. The second part of this inequality is trivially satisfied, so we are left with proving the existence of an index $m \in S'$ such that $\tilde{f}_{m-1} > \tilde{f}_m$.

Divide $\tilde{f}_i$'s, $i = 1, \ldots, s$, into the groups of equals, i.e. $\tilde{f}_i$ and $\tilde{f}_j$, $i, j = 1, \ldots, s$, $i \neq j$, belong to the same group if and only if $\tilde{f}_i = \tilde{f}_j$. Our aim is to prove that if $i \in S'$ then every $j \leqslant i$ such that $\tilde{f}_i = \tilde{f}_j$ also belongs to S'. Indeed, since

$$\tilde{f}_j - c_j = \tilde{f}_i - c_j \leqslant \tilde{f}_i - c_i, \tag{31}$$

and

$$k_j - \tilde{f}_j = k_j - \tilde{f}_i \geqslant k_i - \tilde{f}_i, \tag{32}$$

we have that $j \in S'$. Finally, among all indices that belong to S' choose the minimal ones in the groups of equals $\tilde{f}_i$'s. Denote this set by $\overline{S}$, it is nonempty and any index $m \in \overline{S}$ satisfies all required properties. This finishes our proof. □

5. Main result – connections of the second type

In this section, we consider the series connections of the second type of three or more linear systems. In Theorem 7 we give a complete solution to the Problem 1.

5.1. The controllable case

Theorem 4. *Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \ldots, n$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$ be such that the pair (A_1, B_1) is controllable with $c_1 \geqslant \cdots \geqslant c_s > 0$ as controllability indices. Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers, $\sum_{i=1}^{s} k_i = \sum_{i=1}^{n} n_i$. There exist matrices $X_{ij} \in \mathbb{F}^{n_{i+1} \times n_j}$, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & \ddots & 0 & 0 & B_1 \\ X_{11} & A_2 & 0 & \ddots & 0 & 0 \\ X_{21} & X_{22} & A_3 & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ X_{n-11} & X_{n-12} & \ddots & X_{n-1n-1} & A_n & 0 \end{array}\right] \tag{33}$$

is controllable, and has $k_1 \geqslant \cdots \geqslant k_s$ as its controllability indices if and only if the following two conditions are valid

(i) $k_i \geqslant c_i, \quad i = 1, \ldots, s,$

(ii) $(k_1 - c_1, \ldots, k_s - c_s) \prec \left(\sum_{i=2}^{n} d(\alpha^i_{n_i}), \ldots, \sum_{i=2}^{n} d(\alpha^i_{n_i-s+1}) \right),$

where $\alpha^i_1 | \cdots | \alpha^i_{n_i}$ are the invariant factors of $\lambda I - A_i$, $i = 1, \ldots, n$, and $s = \operatorname{rank} B_1$.

Note that if r_i is the number of nontrivial invariant factors of $\lambda I - A_i$, $i = 1, \ldots, n$, then from the condition (ii) we have

$$s \geqslant r_i, \quad i = 1, \ldots, n.$$

Proof. *Necessity*: Suppose that there exist matrices X_{ij}, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix (33) has the wanted properties. Denote by $f^j_1 \geqslant \cdots \geqslant f^j_s > 0$ the controllability indices of the submatrices of (33) formed by its first $n_1 + \cdots + n_{j+1}$, $j = 0, \ldots, n-1$, rows. Here, $f^0_i = c_i$ and $f^{n-1}_i = k_i$, $i = 1, \ldots, s$. By Theorem 3, we obtain the following conditions:

$$f^{i-1}_j \geqslant f^{i-2}_j, \quad j = 1, \ldots, s, \tag{34}$$

$$(f^{i-1}_1 - f^{i-2}_1, \ldots, f^{i-1}_s - f^{i-2}_s) \prec (d(\alpha^i_{n_i}), \ldots, d(\alpha^i_{n_i-s+1})), \tag{35}$$

for every $i = 2, \ldots, n$.

So, by unifying these conditions, and by Lemma 4, we obtain:

$$k_i \geqslant c_i, \quad i = 1, \ldots, s,$$

$$(k_1 - c_1, \ldots, k_s - c_s) \prec \left(\sum_{i=2}^{n} d(\alpha^i_{n_i}), \ldots, \sum_{i=2}^{n} d(\alpha^i_{n_i-s+1}) \right),$$

as wanted.

Sufficiency: Let the conditions (i) and (ii) be valid. Our aim is to prove that there exist matrices X_{ij}, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix (33) is controllable with $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices. The proof will go by induction. The base of induction is the case $n = 2$, which is already solved by Theorem 3.

Let $n > 2$. Suppose that there exists a nonincreasing sequence of positive integers $f_1 \geqslant \cdots \geqslant f_s$ such that:

$$f_i \geqslant c_i, \quad i = 1, \ldots, s, \tag{36}$$

$$(f_1 - c_1, \ldots, f_s - c_s) \prec \left(\sum_{i=2}^{n-1} d(\alpha^i_{n_i}), \ldots, \sum_{i=2}^{n-1} d(\alpha^i_{n_i - s+1}) \right), \tag{37}$$

and

$$k_i \geqslant f_i, \quad i = 1, \ldots, s, \tag{38}$$

$$(k_1 - f_1, \ldots, k_s - f_s) \prec (d(\alpha^n_{n_n}), \ldots, d(\alpha^n_{n_n - s+1})). \tag{39}$$

Then, from the induction hypothesis, there exist matrices X_{ij}, $1 \leqslant j \leqslant i \leqslant n-2$, such that the submatrix formed by the first $\sum_{i=1}^{n-1} n_i$ rows of (33) is controllable, and has $f_1, \ldots, f_s$ as controllability indices. Thus, by applying Theorem 3, there exist matrices $X_{n-1,j}$ $1 \leqslant j \leqslant n-1$, such that the matrix (33) is controllable with $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices.

So, we are left with proving the existence of $f_1 \geqslant \cdots \geqslant f_s$ which satisfy (36)–(39).

Let $\beta(x) := \sum_{i=2}^{n-1} d(\alpha^i_{n_i - x+1})$ and $\alpha(x) := d(\alpha^n_{n_n - x+1})$, $x = 1, \ldots, s$. Then the condition (ii) becomes

$$(k_1 - c_1, \ldots, k_s - c_s) \prec (\alpha(1) + \beta(1), \ldots, \alpha(s) + \beta(s)).$$

By Lemma 5, there exist nonincreasing, positive integers $f_1 \geqslant \cdots \geqslant f_s$ such that

$$k_i \geqslant f_i \geqslant c_i, \quad i = 1, \ldots, s,$$
$$(f_1 - c_1, \ldots, f_s - c_s) \prec (\beta(1), \ldots, \beta(s)),$$
$$(k_1 - f_1, \ldots, k_s - f_s) \prec (\alpha(1), \ldots, \alpha(s)),$$

which concludes our proof. □

5.2. *The noncontrollable case*

Now we pass to a new problem. The following step is to solve the previous problem in the noncontrollable case, i.e., when the pair (A_1, B_1) and the matrix (7) are not controllable. First, we give Theorem 5, which is in some sense weaker than Theorem 5.1 from [1], but however more useful in questions of prolongation:

Theorem 5. *Let $\mathbb{F}$ be an algebraically closed field. Let $A_1 \in \mathbb{F}^{n_1 \times n_1}$, $B_1 \in \mathbb{F}^{n_1 \times m_1}$, $A_2 \in \mathbb{F}^{n_2 \times n_2}$ be such that the pair (A_1, B_1) is controllable with $c_1 \geqslant \cdots \geqslant c_s > 0$ as controllability indices. Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers, such that $\sum_{i=1}^{s} k_i \leqslant n_1 + n_2$. There exists a matrix $X_1 \in \mathbb{F}^{n_2 \times n_1}$ such that*

$$\begin{bmatrix} A_1 & 0 & B_1 \\ X_1 & A_2 & 0 \end{bmatrix} \tag{40}$$

has $k_1 \geqslant \cdots \geqslant k_s$ *as controllability indices* (*the dimension of the noncontrollable block is* $d = n_1 + n_2 - \sum_{i=1}^{s} k_i$) *if and only if the following conditions are valid*:

(i) $k_i \geqslant c_i, \quad i = 1, \ldots, s,$

(ii) $\left(k_1 - c_1, \ldots, k_s - c_s, \underbrace{1, \ldots, 1}_{d}\right) \prec (A^1, \ldots, A^{s+d}),$

where $A^i = d(\alpha_{n_2-i+1}), i = 1, \ldots, s + d$, *and* $\alpha_1 | \cdots | \alpha_{n_2}$ *are the invariant factors of* $\lambda I - A_2$.

Proof. Without loss of generality, consider the pair (A_1, B_1) in its Brunovsky canonical form (A_c, B_c). Sufficiency of the conditions (i)–(ii) can be proved analogously as the sufficiency part of the proof of Theorem 4.1 from [1] i.e., by using Lemma 2 and Lemma 3. These lemmas, together with the condition (ii), allow us to put the matrix A_2, by similarity operations, into a block lower triangular form, such that the blocks below the principal diagonal have nonzero entries only in their last rows. On the principal diagonal it has blocks in the form of companion matrices, the first d of them of dimension 1 and the following s of them of dimension $k_i - c_i$, $i = 1, \ldots, s$, respectively.

Let $Q := \{i \in \{1, \ldots, s\} | k_i \neq c_i\}$. Define the matrix X_1 such that it has zeros under the block corresponding to c_i in A_c, for all $i \in \{1, \ldots, s\} \setminus Q$. Now put units in the matrix X_1 at the positions $(d + \sum_{i=1}^{j} (k_i - c_i), 1 + \sum_{i=1}^{j-1} c_i)$, $j \in Q$, and all other entries in the matrix X_1 put to be zeros. Then the matrix (40) has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices and d is the dimension of its noncontrollable part.

Now, we can proceed with the necessity part of the proof. Let f_i^j be the ith row of the identity matrix I_j. Let

$$\left[\begin{array}{c|c|c|c|c} K & 0 & 0 & 0 & J \\ \hline 0 & M & 0 & T & 0 \\ \hline S & 0 & N & 0 & 0 \end{array}\right], \tag{41}$$

where

$$J := \mathrm{diag}(\underbrace{e_{x+1}^{x+1}, \ldots, e_{x+1}^{x+1}}_{d}), \quad S := \mathrm{diag}(\underbrace{f_1^{x+1}, \ldots, f_1^{x+1}}_{d}),$$

$$T := \mathrm{diag}(e_{k_1}^{k_1}, \ldots, e_{k_s}^{k_s}), \quad K := \mathrm{diag}(\underbrace{L, \ldots, L}_{d}), \quad M := \mathrm{diag}(U_1, \ldots, U_s),$$

where $L := C(\lambda^{x+1})$, $U_i := C(\lambda^{k_i})$, $i = 1, \ldots, s$ and $x := \max\{k_1, c_1\}$. Also, the matrix N has the same invariant polynomials as the matrix (40).

Thus, the matrix (41) is controllable and has

$$\underbrace{x + 2 \geqslant \cdots \geqslant x + 2}_{d} \geqslant k_1 \geqslant \cdots \geqslant k_s$$

as controllability indices. The submatrix

$$\begin{bmatrix} M & 0 & T \\ 0 & N & 0 \end{bmatrix}$$

of (41), has the same feedback invariants as the matrix (40), i.e., they are feedback equivalent. So, there exists an invertible matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \in \mathbb{F}^{(n_1+n_2)\times(n_1+n_2)}$ such that the matrix (41) is feedback equivalent to the following one

$$\left[\begin{array}{c|c|c|c|c} K & 0 & 0 & J & 0 \\ \hline P_2S & A_1 & 0 & 0 & B_1 \\ \hline P_4S & X_1 & A_2 & 0 & 0 \end{array}\right]. \tag{42}$$

Furthermore, its submatrix

$$\left[\begin{array}{c|c|c|c|c} K & 0 & 0 & J & 0 \\ \hline P_2S & A_1 & 0 & 0 & B_1 \end{array}\right]$$

is controllable and has $\underbrace{x+1 \geqslant \cdots \geqslant x+1}_{d} \geqslant c_1 \geqslant \cdots \geqslant c_s$ as controllability indices. Thus, if denote by

$$\overline{A}_1 := \begin{bmatrix} K & 0 \\ P_2S & A_1 \end{bmatrix}, \quad \overline{B}_1 := \begin{bmatrix} J & 0 \\ 0 & B_1 \end{bmatrix}, \quad \overline{X} := \begin{bmatrix} P_4S & X_1 \end{bmatrix},$$

we have that the pair $(\overline{A}_1, \quad \overline{B}_1)$ is controllable with $\underbrace{x+1 \geqslant \cdots \geqslant x+1}_{d} \geqslant c_1 \geqslant \cdots \geqslant c_s$ as controllability indices. Also, $\overline{X}$ is such that the matrix

$$\begin{bmatrix} \overline{A}_1 & 0 & \overline{B}_1 \\ \overline{X} & A_2 & 0 \end{bmatrix} \tag{43}$$

is controllable with $\underbrace{x+2 \geqslant \cdots \geqslant x+2}_{d} \geqslant k_1 \geqslant \cdots \geqslant k_s$ as controllability indices. Finally, by applying Theorem 3, we obtain:

(i) $k_i \geqslant c_i, \quad i = 1, \ldots, s,$

(ii) $\left(k_1 - c_1, \ldots, k_s - c_s, \underbrace{1, \ldots, 1}_{d}\right) \prec (A^1, \ldots, A^{s+d}),$

as wanted. □

Now, we consider the case when the matrix pair (A_1, B_1) is not controllable. The result is given in the following theorem:

Theorem 6. *Let $\mathbb{F}$ be an algebraically closed field. Let $A_1 \in \mathbb{F}^{n_1\times n_1}$, $B_1 \in \mathbb{F}^{n_1\times m_1}$, $A_2 \in \mathbb{F}^{n_2\times n_2}$ be such that the pair (A_1, B_1) has $c_1 \geqslant \cdots \geqslant c_s > 0$ as controllability indices ($n_1 - \sum_{i=1}^{s} c_i = \delta \geqslant 0$). Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers, $\sum_{i=1}^{s} k_i \leqslant n_1 + n_2$. There exists a matrix $X_1 \in \mathbb{F}^{n_2\times n_1}$ such that*

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ X_1 & A_2 & 0 \end{array}\right] \tag{44}$$

has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices (the dimension of the noncontrolable block is $d = n_1 + n_2 - \sum_{i=1}^{s} k_i$) if and only if the following conditions are valid:

(i) $d \geqslant \delta$,

(ii) $k_i \geqslant c_i, \quad i = 1, \ldots, s$,

(iii) $\left(k_1 - c_1, \ldots, k_s - c_s, \underbrace{1, \ldots, 1}_{d-\delta}\right) \prec (A^1, \ldots, A^{d-\delta+s})$,

where $A^i = d(\alpha_{n_2-i+1}), i = 1, \ldots, d-\delta+s$, *and* $\alpha_1 | \cdots | \alpha_{n_2}$ *are the invariant factors of* $\lambda I - A_2$.

Proof. *Sufficiency*: Let

$$\begin{bmatrix} N_1 & 0 & 0 \\ * & M_1 & M_2 \end{bmatrix}$$

be a Kalman decomposition of the matrix $[A_1 \quad B_1]$. Here the pair (M_1, M_2) is controllable with $c_1 \geqslant \cdots \geqslant c_s$ as controllability indices and $N_1 \in \mathbb{F}^{\delta \times \delta}$. Thus, the matrix (44) is feedback equivalent to the following one

$$\left[\begin{array}{c|c|c|c} N_1 & 0 & 0 & 0 \\ \hline * & M_1 & 0 & M_2 \\ \hline X_{11} & X_{12} & A_2 & 0 \end{array}\right], \tag{45}$$

where $X_{11} \in \mathbb{F}^{n_2 \times \delta}$, $X_{12} \in \mathbb{F}^{n_2 \times (n_1-\delta)}$ and $*$ denotes unimportant entries.

Our aim is to define matrices X_{11} and X_{12} such that the matrix (45) has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices and d as the dimension of the noncontrollable part.

Now, from the conditions (i)–(iii) and by applying Theorem 5, we have that there exists a matrix $X_{12} \in \mathbb{F}^{n_2 \times (n_1-\delta)}$ such that the matrix

$$\begin{bmatrix} M_1 & 0 & M_2 \\ X_{12} & A_2 & 0 \end{bmatrix} \tag{46}$$

has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices, while its noncontrollable part is of the dimension $d - \delta$.

Now, the matrix (45) becomes

$$\left[\begin{array}{c|c|c|c} N_1 & 0 & 0 & 0 \\ \hline * & M_1 & 0 & M_2 \\ \hline 0 & X_{12} & A_2 & 0 \end{array}\right]. \tag{47}$$

Put the submatrix (46) of (47) into its Kalman decomposition form. Now, the matrix (47) is feedback equivalent to the matrix

$$\left[\begin{array}{c|c|cc} N_1 & 0 & 0 & 0 \\ * & N_2 & 0 & 0 \\ \hline * & * & \overline{M_1} & \overline{M_2} \end{array}\right] \tag{48}$$

where $(\overline{M_1}, \overline{M_2})$ is controllable pair with $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices and $N_2 \in \mathbb{F}^{(d-\delta)\times(d-\delta)}$. Thus, the dimension of the noncontrollable part of the matrix (48) is d, as wanted.

Necessity: To prove the necessity of the conditions, suppose that there exists a matrix X_1 such that the matrix (44) has the required properties. As in the sufficiency part of the proof, it can be shown that the matrix (44) is feedback equivalent to the matrix (45). Thus, there exist matrices X_{11} and X_{12} such that the matrix (45) has wanted properties. Let $f_1 \geqslant \cdots \geqslant f_s$ be controllability indices and $d - \delta' \geqslant 0$ be dimension of the noncontrollable part of the matrix

$$\begin{bmatrix} M_1 & 0 & M_2 \\ X_{12} & A_2 & 0 \end{bmatrix}. \tag{49}$$

By applying Theorem 5, we obtain the following conditions:

(i) $d \geqslant \delta'$,

(ii) $f_i \geqslant c_i, \quad i = 1, \ldots, s$,

(iii) $\left(f_1 - c_1, \ldots, f_s - c_s, \underbrace{1, \ldots, 1}_{d-\delta'} \right) \prec (A^1, \ldots, A^{d-\delta'+s})$.

Now, the matrix (45) is feedback equivalent to the matrix of the form (48), where the pair $(\overline{M}_1, \overline{M}_2)$ is controllable with $f_1 \geqslant \cdots \geqslant f_s$ as controllability indices and $N_2 \in \mathbb{F}^{(d-\delta')\times(d-\delta')}$. Thus, we have $f_i = k_i$, $i = 1, \ldots, s$, and $\delta = \delta'$, which concludes our proof. $\square$

Finally, we give the solution for the problem of series connection of the second type of arbitrarily many linear systems in the noncontrollable case:

Theorem 7. *Let $\mathbb{F}$ be an algebraically closed field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \ldots, n$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$ be such that a pair (A_1, B_1) has $c_1 \geqslant \cdots \geqslant c_s > 0$ as controllability indices ($n_1 - \sum_{i=1}^{s} c_i = \delta \geqslant 0$). Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers, such that $\sum_{i=1}^{s} k_i \leqslant \sum_{i=1}^{n} n_i$. There exist matrices $X_{ij} \in \mathbb{F}^{n_{i+1} \times n_j}$, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix*

$$\left[\begin{array}{cccc|c} A_1 & & & 0 & B_1 \\ X_{11} & A_2 & & & 0 \\ \ddots & \ddots & \ddots & & \vdots \\ X_{n-11} & \ddots & X_{n-1n-1} & A_n & 0 \end{array}\right] \tag{50}$$

has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices (the dimension of the noncontrollable block is $d = \sum_{i=1}^{n} n_i - \sum_{i=1}^{s} k_i \geqslant 0$) if and only if the following conditions are valid:

(i) $d \geqslant \delta$,

(ii) $k_i \geqslant c_i, \quad i = 1, \ldots, s$,

(iii) $\left(k_1 - c_1, \ldots, k_s - c_s, \underbrace{1, \ldots, 1}_{d-\delta} \right) \prec \left(\sum_{i=2}^{n} A_1^i, \ldots, \sum_{i=2}^{n} A_{d-\delta+s}^i \right)$,

where $A_i^j = d(\alpha_{n_j - i+1}^j)$, $i = 1, \ldots, d - \delta + s$, and $\alpha_1^j | \cdots | \alpha_{n_j}^j$ are the invariant factors of $\lambda I - A_j$, $j = 2, \ldots, n$.

Proof. *Necessity*: Suppose that there exist matrices $X_{ij} \in \mathbb{F}^{n_{i+1} \times n_j}$, $1 \leqslant j \leqslant i \leqslant n-1$, such that the matrix (50) has wanted properties. Let $f_1^i \geqslant \cdots \geqslant f_s^i > 0$ be the controllability indices and let d_i be the dimension of the noncontrollable part of the submatrix of (50) formed by its first $\sum_{j=1}^{i+1} n_j$ rows, for every $i = 0, \ldots, n-1$. So, we have $c_i = f_i^0$ and $k_i = f_i^{n-1}$, $i = 1, \ldots, s$, $d_0 = \delta$ and $d_{n-1} = d$.

By Theorem 6, we obtain the following conditions:

$$d_i \geqslant d_{i-1}, \tag{51}$$

$$f_j^i \geqslant f_j^{i-1}, \quad j = 1, \ldots, s, \tag{52}$$

$$\left(f_1^i - f_1^{i-1}, \ldots, f_s^i - f_s^{i-1}, \underbrace{1, \ldots, 1}_{d_i - d_{i-1}}\right) \prec (A_1^{i+1}, \ldots, A_{d_i - d_{i-1}+s}^{i+1}), \tag{53}$$

for $i = 1, \ldots, n-1$.

Thus, by unifying the conditions (51)–(53) for all $i = 1, \ldots, n$, and by Lemma 4, we obtain

(i) $d \geqslant \delta$,

(ii) $k_i \geqslant c_i, \quad i = 1, \ldots, s$,

(iii) $\left(k_1 - c_1, \ldots, k_s - c_s, \underbrace{1, \ldots, 1}_{d-\delta}\right) \prec \left(\sum_{i=2}^n A_1^i, \ldots, \sum_{i=2}^n A_{d-\delta+s}^i\right)$,

as wanted.

Sufficiency: Analogously as in the proofs of Theorem 4 and Theorem 6, our aim is to define a nonincreasing sequence of positive integers $f_1 \geqslant \cdots \geqslant f_s$ and a nonnegative number d_1 such that $d \geqslant d_1 \geqslant \delta$, and

$$k_i \geqslant f_i \geqslant c_i, \quad i = 1, \ldots, s,$$

$$\left(f_1 - c_1, \ldots, f_s - c_s, \underbrace{1, \ldots, 1}_{d_1 - \delta}\right) \prec (A(1), \ldots, A(d_1 - \delta + s)),$$

$$\left(k_1 - f_1, \ldots, k_s - f_s, \underbrace{1, \ldots, 1}_{d - d_1}\right) \prec (B(1), \ldots, B(d - d_1 + s)),$$

where $A(i) := A_i^2$, and $B(i) := \sum_{i=3}^n A_i^j$, $i = 1, \ldots, d - \delta + s$.

Furthermore, if $k_i = c_i$ for some $i \in \{1, \ldots, s\}$, then we define $f_i := k_i = c_i$. So, we can "cut off" those indices. Thus, we can assume that $k_i > c_i, i = 1, \ldots, s$. Also, define $c_i = 1$ and $k_i = 2$, for $i = s+1, \ldots, s+d-\delta$.

Now, the condition (iii) becomes

$$(k_1 - c_1, \ldots, k_{s+d-\delta} - c_{s+d-\delta}) \prec (A(1) + B(1), \ldots, A(d-\delta+s) + B(d-\delta+s)).$$

By Lemma 5, we have that there exist $f_1 \geqslant \cdots \geqslant f_{s+d-\delta} > 0$ such that

$$k_i \geqslant f_i \geqslant c_i, \quad i = 1, \ldots, s+d-\delta,$$

$$(f_1 - c_1, \ldots, f_{s+d-\delta} - c_{s+d-\delta}) \prec (A(1), \ldots, A(d-\delta+s)),$$

$$(k_1 - f_1, \ldots, k_{s+d-\delta} - f_{s+d-\delta}) \prec (B(1), \ldots, B(d-\delta+s)).$$

For $i = s+1, \ldots, s+d-\delta$, we have

$$2 \geqslant f_i \geqslant 1.$$

If $f_i = 1, i = s+1, \ldots, s+d-\delta$, let $m := 0$. Otherwise, let

$$m := \max\{i \mid f_{s+i} = 2\}, \quad d - \delta \geqslant m > 0.$$

Then

$$\Big(f_1 - c_1, \ldots, f_s - c_s, \underbrace{1, \ldots, 1}_{m}\Big) \prec (A(1), \ldots, A(m+s)),$$

$$\Big(k_1 - f_1, \ldots, k_s - f_s, \underbrace{1, \ldots, 1}_{l}\Big) \prec (B(1), \ldots, B(l+s)),$$

where $m + l = d - \delta$.

Thus, let $d_1 := d - l = \delta + m$. This concludes our proof. □

6. Connections of the first type – sufficient conditions

In this section we deal with the series connections of the first type of arbitrarily many linear systems, i.e., we study the properties of the matrix

$$\left[\begin{array}{cccccc|c} A_1 & 0 & 0 & \cdots & 0 & & B_1 \\ X_1 & A_2 & 0 & \cdots & 0 & & 0 \\ 0 & X_2 & A_3 & \cdots & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & 0 & \ddots & X_{n-1} & A_n & & 0 \end{array}\right], \tag{54}$$

when matrices $X_1, \ldots, X_{n-1}$ vary.

The problem of determining the controllability of the matrix (54) is solved in [3]. However, the problem of determining the possible controllability indices of the matrix (54) when matrices $X_1, \ldots, X_{n-1}$ vary, still remains open. In the following theorem, we present the sufficient conditions for it. This result gives the solution to Problem 2.

Theorem 8. *Let $\mathbb{F}$ be an algebraically closed field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \ldots, n$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$, rank $B_1 = s$, be matrices such that the pair (A_1, B_1) is controllable with $c_1 \geqslant \cdots \geqslant c_s > 0$ as its controllability indices. Let $k_1 \geqslant \cdots \geqslant k_s > 0$ be positive integers. Let $S = \{i \mid k_i \neq c_i\} \subset \{1, \ldots, s\}$ and let $r = \sharp S$.*

If

(o) $n_i \geqslant r, \quad i = 1, \ldots, n,$

(i) $k_i \geqslant c_i, \quad i = 1, \ldots, s,$

(ii) $(k_1 - c_1, \ldots, k_s - c_s) \prec \left(\sum_{i=2}^{n-1} \overline{D}_1^i + D_1^n, \ldots, \sum_{i=2}^{n-1} \overline{D}_s^i + D_s^n\right),$

then there exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$, $i = 1, \ldots, n-1$, such that the matrix (54) is controllable and has $k_1 \geqslant \cdots \geqslant k_s > 0$ as controllability indices.

Here $D_i^j = d(\alpha^j_{n_j-i+1})$, $i = 1, \dots, s$, *and* $\alpha^i_1 | \cdots | \alpha^i_{n_i}$ *are the invariant factors of* $\lambda I - A_i$, r_i *of them nontrivial*, $i = 2, \dots, n$. *Also*, $(\overline{D}^j_1, \dots, \overline{D}^j_s)$ *is the maximal (in the sense of majorization) nonincreasing partition majorated by the partition* $(D^j_1, \dots, D^j_s)$ *such that* $\overline{D}^j_i > 0$ *for every* $i = 1, \dots, r$, *and for each* $j = 2, \dots, n-1$.

Remark 9. Let $j \in \{2, \dots, n-1\}$. Then $(\overline{D}^j_1, \dots, \overline{D}^j_s)$ satisfy the following properties:

$$(\overline{D}^j_1, \dots, \overline{D}^j_s) \prec (D^j_1, \dots, D^j_s),$$

and for every nonincreasing partition $(x_1, \dots, x_s)$, such that

$$(x_1, \dots, x_s) \prec (D^j_1, \dots, D^j_s),$$

and such that $x_i > 0$, $i = 1, \dots, r$, we have

$$(x_1, \dots, x_s) \prec (\overline{D}^j_1, \dots, \overline{D}^j_s).$$

Since $\sum_{i=1}^s D^j_i = n_j \geqslant r$, $\overline{D}^j_i$ with these properties always exist. In particular, we can define them explicitly as follows: if $D^j_r > 0$, then $\overline{D}^j_i := D^j_i$, $i = 1, \dots, s$. If $D^j_r = 0$ then let $b = \max\{a < r \mid \sum_{i=a+1}^r D^j_i \geqslant r - a\}$, and define

$$\begin{aligned}
\overline{D}^j_i &:= D^j_i, \quad i = 1, \dots, b,\\
\overline{D}^j_{b+1} &:= \sum_{i=b+1}^{r} D^j_i - r + b + 1,\\
\overline{D}^j_i &:= 1, \quad i = b+2, \dots, r,\\
\overline{D}^j_i &:= 0, \quad i = r+1, \dots, s.
\end{aligned}$$

Proof. Put the pair (A_1, B_1) in its Brunovsky canonical form (A_c, B_c). Note that from the condition (ii) follows $r \geqslant \max_{i=1,\dots,n}\{r_i\}$, and hence $\overline{D}^j_i = 0$, $j = 2, \dots, n-1$, and $D^n_i = 0$ for every $i > s$.

Let $k_i > c_i$, $i = 1, \dots, s$, i.e. let $r = s$. From the conditions (i) and (ii), and by applying Lemma 5 repeatedly, we obtain that there exist positive integers $f^i_1 \geqslant \cdots \geqslant f^i_s$, $i = 1, \dots, n-2$, such that

$$\begin{aligned}
&k_i \geqslant f_i^{n-2} \geqslant f_i^{n-3} \geqslant \cdots \geqslant f_i^1 \geqslant c_i, \quad i = 1, \dots, s,\\
&(f_1^1 - c_1, \dots, f_s^1 - c_s) \prec (\overline{D}^2_1, \dots, \overline{D}^2_s),\\
&(f_1^2 - f_1^1, \dots, f_s^2 - f_s^1) \prec (\overline{D}^3_1, \dots, \overline{D}^3_s),\\
&\dots\\
&(f_1^{n-2} - f_1^{n-3}, \dots, f_s^{n-2} - f_s^{n-3}) \prec (\overline{D}^{n-1}_1, \dots, \overline{D}^{n-1}_s),\\
&(k_1 - f_1^{n-2}, \dots, k_s - f_s^{n-2}) \prec (D^n_1, \dots, D^n_s).
\end{aligned}$$

Since $\overline{D}^j_i \geqslant 1$, $i = 1, \dots, s$, $j = 2, \dots, n-1$, we obtain that $f^1_i - c_i \geqslant 1$ and $f^j_i - f^{j-1}_i \geqslant 1$, $i = 1, \dots, s$, $j = 2, \dots, n-2$. Like in [1], since $\mathbb{F}$ is algebraically closed field, we can put the matrices A_j into the following similar forms:

$$P_jA_jP_j^{-1} = C^j = \begin{bmatrix} C_1^j & & & 0 \\ E_{11}^j & C_2^j & & \\ \ddots & \ddots & \ddots & \\ E_{s-11}^j & \cdots & E_{s-1s-1}^j & C_s^j \end{bmatrix},$$

$j = 2, \ldots, n$. Here C_i^j has the form of a companion matrix of the dimension $f_i^{j-1} - f_i^{j-2}$, $j = 2, \ldots, n$, $f_i^0 = c_i$, $f_i^{n-1} = k_i$, $i = 1, \ldots, s$, and matrices $E_{ik}^j \in \mathbb{F}^{(f_i^{j-1}-f_i^{j-2})\times(f_k^{j-1}-f_k^{j-2})}$, $1 \leqslant k \leqslant i \leqslant s-1$, $j = 2, \ldots, n$, have all entries equal to zero except the ones in the last row.

Put units in the matrix P_2X_1 at the entries $(\sum_{i=1}^{j}(f_i^1 - c_i), 1 + \sum_{i=1}^{j-1} c_i)$ for $j = 1, \ldots, s$, and all other entries in the matrix P_2X_1 put to be zero. Then the matrix

$$\begin{bmatrix} A_c & 0 & B_c \\ P_2X_1 & P_2A_2P_2^{-1} & 0 \end{bmatrix} \tag{55}$$

is controllable and $f_1^1 \geqslant \cdots \geqslant f_s^1$ are its controllability indices ($f_i^1 \geqslant c_i + 1$, $i = 1, \ldots, s$).

Furthermore, put units in the matrices $P_kX_{k-1}P_{k-1}^{-1}$, $k = 3, \ldots, n$, at the entries $(\sum_{i=1}^{j}(f_i^{k-1} - f_i^{k-2}), 1 + \sum_{i=1}^{j-1}(f_i^{k-2} - f_i^{k-3}))$, $j = 1, \ldots, s$, and all other entries in the matrices $P_kX_{k-1}P_{k-1}^{-1}$, $k = 3, \ldots, n$, put to be zero. Obviously, such obtained matrix

$$\begin{bmatrix} A_c & 0 & & \ddots & 0 & B_c \\ P_2X_1 & C^2 & \ddots & & \ddots & 0 \\ 0 & P_3X_2P_2^{-1} & C^3 & \ddots & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 0 & P_nX_{n-1}P_{n-1}^{-1} & C^n & 0 \end{bmatrix} \tag{56}$$

is controllable and has $k_1 \geqslant \cdots \geqslant k_s$ as controllability indices, as wanted.

If there exists $i \in \{1, \ldots, r\}$ such that $k_i = c_i$, then put zeros in the matrix X_1 under the block corresponding to c_i in A_c. In such a way we reduce the problem to the case $s = r$. $\square$

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